Simple Models of Flat Fading and Cross-Correlation in Space
Diversity and MIMO Systems

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Abstract - This paper presents new equations and BER curves obtained via Monte Carlo simulation for multiple input/output (MISO and MIMO) systems affected by flat but near-to-fast fading and channel cross-correlation. The degradation of performance in respect to the idealized state without correlation and fading is evaluated and the existence of an irreducible error zone is justified.

Index Terms – MIMO systems, flat fading, cross-correlation, MT simulation, Alamouti code

1. Introduction

Over the last decade, space diversity and MIMO systems have gained attention both from research and industrial communities. These techniques offer a substantial increase of a link capacity and reveal a high resistance to channel fading. However, to take advantage of this opportunity, some important conditions must be satisfied. Among others, the channels shall be principally orthogonal, the propagation medium shall respond to a single-path and a time-invariant model and the noise and interference shall co-operate with the AWGN format. Physical channels usually do not fulfill these conditions. The question is what are the real benefits of using the multi-antenna systems?

A multi-path fading channel is generally characterized as a linear, time-varying medium having a time-frequency response \( c(t,f) \), which is a wide sense stationary random process in the time domain. Time variations of \( c(t,f) \) result in frequency spreading, while multi-path propagation reveals as the time spreading of a transmitted signal. The first phenomenon is expressed by the Doppler shift \( B_d \), while the second one –
by coherence bandwidth $B_c = 1/T_m$, where $T_m$ is a mean square delay of the multi-path signal. Consequently, the channel can be viewed as a double spread quantity, which produces either selective or non-selective fades, both fast and slow, depending on the ratios of $B_d$ and $B_c$ to the transmission rate $1/T$, Table 1. We will focus on the non-selective or flat fading, because any selective one can be converted into a flat one thanks to the possibility of orthogonal frequency division application (OFDM). We will, however, confine our discussion to slow and near - to - fast fading.

<table>
<thead>
<tr>
<th>Coherence $B_c$</th>
<th>Doppler bandwidth $B_d$</th>
<th>$B_d &gt;&gt; 1/T$</th>
<th>$B_d &lt;&lt; 1/T$</th>
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<td>$B_c &gt;&gt; 1/T$</td>
<td>Non-selective (flat) and slow fading</td>
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<td>$B_c &lt;&lt; 1/T$</td>
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Table 1 Classes of fading versus its data to transmission rate 1/T

Suppose, the signal bandwidth is $W \approx 1/T << B_c$, ($B_c << 1/T$). Then all the frequency components of the signal $s(f)\exp(j2\pi ft)$ undergo the same attenuation and phase shift. The received signal $r(t)$ can be expressed by the following equation for the noise absence case [1]:

$$r(t) = \int_{-\infty}^{\infty} c(t; f)s(f)e^{j2\pi ft} df = c(t; 0)\int_{-\infty}^{\infty} s(f)e^{j2\pi ft} df = a(t)e^{j\theta(t)}S(t)$$  \hspace{1cm} (1)$$

where, by definition, $c(t,0) = a(t)\exp[j\theta(t)]$, $a(t)$ represents the envelope and $\theta(t)$ – the phase of the equivalent channel response; $S(t)$ – the low-pass signal. Thus, a flat fading channel has a **time-varying multiplicative effect** on the transmitted signal. The crucial point is, however, how great is this impact on the reception process and what the possible counteractions are. The aim of this paper is to develop simple, but content-oriented models of the flat fading and channel correlation, including the floor effect. These models allow to assess the losses of BER in real environments.

The main contribution of this paper consists in releasing the usually taken assumptions that the transfer function of a channel varies slowly, i.e. $h(t) \approx h(t+T)$ and that the cross-correlation between channels does not exist. The presented approach is very simple and therefore useful.
The paper is composed of four following sections: (2) Very slow flat fading and switching diversity, (3) Slow flat fading and combining diversity, (4) Real flat fading and floor effect, (5) Channel correlation in MIMO systems.

2. Very slow flat fading and switching diversity

Very slow flat fading stands here for the Doppler bandwidth, $B_d<<1/T$ and $B_c>>1/T$. To illustrate the effect of this phenomenon on BER curves the set of $N$ orthogonal Rayleigh channels with known $SNR=\gamma$ is considered. The best channel of the highest $\gamma$ is currently used for transmission. The distribution of $\gamma$ for the chosen channel $n$ out of $N$ is given by the positional statistics \[ f(\gamma/n, N) = \frac{N!\gamma_0}{(n-1)!(N-n)!} F^{n-1}(\gamma)[1-F(\gamma)]^{N-n} \] where $F(\gamma)$ – cumulative distribution function of $\gamma$.

The Rayleigh distribution undergoes - within the power variable - into exponential distribution, \[ f(\gamma) = \gamma_0^m \exp(-\gamma/\gamma_0), \] where $\gamma_0$ is a mean value of $\gamma$. Hence (2) converts to (3):

\[ f(\gamma/n, N) = \frac{n}{\gamma_0} \binom{N}{n} (1-e^{-\gamma/\gamma_0})^{\gamma_0} (e^{-\gamma/\gamma_0})^{N-n} \]

Expanding the binomial $(1-e^{-\gamma/\gamma_0})^{n-1}$ into a sum we obtain finally

\[ f(\gamma/n, N) = \frac{N-n+1}{\gamma_0} \binom{N}{n} \sum_{k=0}^{N-n} (-1)^k \binom{N-n}{k} e^{-\gamma(k+n)/\gamma_0} \]

The probability of error for a typical DPSK transmission is

\[ P(\gamma) = 1/2e^{-\gamma} \] (5)

Because the changes of $\gamma$ are very slow we can calculate the resultant BER as a mean of (5) in respect to the distribution density function (4). The outcome of the integration is

\[ P(\gamma_0/n, N) = \frac{N-n+1}{2} \binom{N}{n} \sum_{k=0}^{N-n} (-1)^k \binom{N-n}{k} \frac{1}{\gamma_0+k+n} \] (6)

A similar averaging of (5) in respect to \[ f(\gamma) = \gamma_0^m \exp(-\gamma/\gamma_0) \] gives

\[ P(\gamma_0) = \frac{1}{2\gamma_0 + 2} \]

The resultant BER curves for all the considered channels are shown in Fig. 1. We can see that for the best channel $n=1$ out of $N=2$ the gain in comparison to a single
Rayleigh channel $N=n=1$ is proportional to SNR and at the level of BER=$10^{-4}$ it reaches $\Delta\text{SNR}=17$ dB. This high gain results from the fact that Rayleigh fading manifests occasionally very deep losses of signal, which is the real cause of errors. If, for example, the probability of errors is on the level of $10^{-2}$ in a single Rayleigh channel, then using two such channels, we obtain the error rate of $10^{-4}$ (Fig.1, dotted lines).

The gain in switched diversity systems increases with a number of channels used. This technique is, however, far from optimum one for $N>>1$ as the most of channels are idle for a long time, while the best one is used. Moreover, the assumption $B_d<<1/T$ is rarely justified.

![Fig. 1. DPSK BER curves for Rayleigh channels with very slow flat fading: $N=1$ – no diversity; $N=2/1$ – two diversity channels, the best used; $N=2/2$ - the second best used](image)

### 3. Slow flat fading and combining diversity

Slow flat fading refers to the case, when $B_d<<1/T$ (and $B_c>>1/T$). A complete system exploiting two transmit antennas and some number of receive antennas and operating in accordance with these assumptions was presented a decade ago by Alamouti [3]. It was assumed that the channel transfer functions respond to the Rayleigh distribution, but within two consecutive symbols they are nearly the same, so $h(t)=h(t+T)$, where $T$ denotes duration. The orthogonal feature of the channels was ensured by using a simple space-time coding (STBC). At the moment of time $t=0$ the signal $s_0$ is sent by the antenna 0 and the signal $s_1$ - by the antenna 1, while at $t=t+T$ the modified signal $-s_1^*$ is sent by the antenna 0 and the signal $s_0^*$ - by the antenna 1, Fig.2. This way the same signal is sent twice
by two antennas operating at the same frequency. Two transmitting and two receiving antennas form the elementary MIMO system. According to Fig.2 the received signals in two steps \( t \) and \( t+T \) are

\[
\begin{align*}
    r_0 &= h_0s_0 + h_1^Ts_1 + n_0 \\
    r_1^T &= -h_0^Ts_1^* + h_1^Ts_0^* + n_1
\end{align*}
\] (8)

The product of a combiner for signal \( s_0 \) is as follows

\[
\tilde{s}_0 = h_0^*r_0 + h_1^*r_1^* = |h_0|^2s_0 + |h_1|^2s_0 + h_1h_0^*s_1 - h_1h_0^*s_1^* + n_0^* + n_1^*
\] (9)

The ML criterion for the phase shift keying signals and for equal \( \text{apriori} \) probabilities is

\[
(\tilde{s} - s_0)(\tilde{s} - s_0)^* \leq (\hat{s} + s_0)(\hat{s} + s_0)^*
\] (10)

where each symbol represents one column vector of size \( \{k\} \).

If the inequality (10) is satisfied the decision is \( s_0 \), otherwise \( s_1 \). The BER curves obtained due to (8-10) are given in Fig.3 and will be discussed later.

4. Real flat fading and the floor effect

The fundamental work on the phenomenon of real flat fading was done several decades ago by Bello and Nelin [4]. Their input equation is

\[
r(t) = Z(t)S(t) + N(t)
\] (11)
where $r(t)$ – the received signal; $S(t)$ – the input signal – a complex time function; $N(t)$ – an additive white Gaussian noise, also a complex function; $Z(t)$ – a multiplicative interference factor having the statistical properties of a complex narrow-band circular symmetric Gaussian process

$$Z(t) = a(t) + jb(t)$$  \hspace{1cm} (12)

where $a$, $b$ – independent Gaussian variables of zero means and variances, $\sigma^2=1/2$. The correlation function of this process can be approximated as follows

$$R(\tau) = 2\sigma^2 \exp(-2\pi B_{0.5} |\tau|)$$  \hspace{1cm} (13)

where $B_{0.5}$ - a half-power bandwidth of the fading spectrum; $\tau$ - a time variable.

The Authors showed that flat fading, even slow enough, can affect the reception process at the level of low error rates. They mathematically justified the existence of the so-called floor effect or irreducible error, which acts independently on noise. The original work is, however, very complex and tied up by many simplifications. The present paper uses a simple digital method of exact Doppler shift \cite{5} and assumes that real flat fading follows this procedure. In the sequel the transmission rate is denoted by $1/T$, the Doppler shift by $B$ and the product $BT$ is called the normalized fading bandwidth. An example of a fading sequence for $BT=0.01$ is given in Fig.3. The left hand side picture shows the phase function $\phi_F$, while the right hand side – its differenced form $\Delta \phi_F = \phi_{F(j+1)} - \phi_{F(j)}$, $j=1,2,...,J$. According to eq.(11) the error in reception of PSK signal - in the absence of noise – takes place if the following inequality is satisfied

$$|\phi_S - \phi_F| > \pi / 2$$  \hspace{1cm} (14)

where $\phi$ is the phase of a useful signal.

We can see from Fig. 3 that there is only one point over the considered 200 samples, when eq.(14) is met. It corresponds to a sudden change of the fading function $Z(t)$, which causes a phase jump from $-\pi$ to $\pi$. Due to the assumed $BT=0.01$, the phase of $Z(t)$ changes - on average - every hundred samples but some of these changes may be soft, as in point 42, while other - abrupt as in point 160. In a single Rayleigh channel the lowest BER is of the order of $BT$, while in diversity one it depends on its order. The complete families of curves for two antennas diversity system, MISO, and four antennas of MIMO system are shown in Fig.4. Original curves of \cite{3} are denoted by bold lines and assigned by $BT=0$. New curves
are assigned by respective normalized bandwidths, BT=0.05 or BT=0.1. The BT=0.1 means that fades change statistically ten times slower than the useful signal.

![Graph](image)

**Fig. 3. An example of phase fading - original function angle[Z(t)] and its differential**

It is evident from Fig.4 that fading, although is flat and relatively slow, affects considerably the process of reception, especially in the transmit diversity case. The new curves, starting from some SNR, run more and more gently and stop to fall at the end. This initiates the so-called irreducible error zone [1]. The losses in comparison to original data [3] are for BT=0.1 and BER=$10^{-2}$ of the order of 6 dB, but at a little lower BER they reach infinity! The respective curve crosses the single Rayleigh channel level (dashed line). It means that there is an additional source of errors, apart from AWGN. Actually, it is the fading source itself.

The error rate for MIMO system holds a quite good level of $10^{-4}$ in an excess of $\Delta$SNR = 6 dB in respect to [3]. The further decrease of BER is, however, blocked by the mentioned floor effect. This was not observed in [3], nor in [6].
Fig. 4. BER curves for two branch diversity and MIMO2x2 systems affected by flat fading

5. Channel correlation in MIMO systems

The direct influence of channel correlation on BER is tackled in [6] and [7]. In [7] a simple method is proposed, which introduces correlation by mixing the response of an individual channel with a response of an artificial reference channel. The constraint $h(t)=h(t+T)$ is, however, still maintained. In the present paper it is released and the curves obtained are shown in Fig.4. They refer to BT=0.1 (upper curve) and BT=0.05 (middle curve). The lowest one plays a role of a reference [3]. The correlation coefficient $\rho=0.5=\text{const.}$

We can see that flat fading at BT=0.1 and mutual channel correlation of $\rho=0.5$ causes joint losses in respect to [3] of the order of 6 dB at BER=$10^{-3}$. Comparing Fig.2 to Fig.3 at the same BER we can deduce that correlation takes nearly a half of this decline (12.5-10=2.5 dB). Some concise approach to these both issues, flat fading and correlation is done in [8]
Fig. 5. BER curves for MIMO2x2 system affected by both fading and channel correlation

6. Conclusion

The aim of this paper is to release the unreal constraints that are placed sometimes on fading phenomena and channel cross-correlation in MISO and MIMO systems. It is shown that flat fading, even slow, shrinks BER to the order of the fading normalized bandwidth BT for a single Rayleigh channel and decreases it by some orders in low rank diversity systems.

The MIMO architecture is more resistant to BER decline. The rate of $10^{-4}$ for MIMO2x2 and $BT=0.1$ is feasible in excess of $\Delta SNR \approx 6$ dB in respect to no fading case. A further decrease of BER is, however, blocked by the floor effect.

The influence of channel cross-correlation is generally much lower than the impact of fading. It is assessed by simulation that down to BER=$10^{-3}$ the MIMO2x2 system experiences correlation losses of merely $\sim 2.5$ dB for $\rho=0.5$.

This paper does not discuss the multi-path selective fading. Readers are encouraged to use OFDM or other countermeasure techniques [9]. There are also more advanced approaches to the considered problems, e.g. [10].

References


