Application of Positional Statistics to BER Formulae Derivation for Switching Diversity MIMO Systems

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Abstract: The closed-form simple expressions for bit error rate (BER) in switching diversity multiple input –multiple output (MIMO) systems operating under flat and quasi-static Rayleigh fading have been derived. This method is based on the positional statistics.
Keywords: BER formulae, Flat fading, MIMO systems, Positional statistics

1. Introduction

The multiple input – multiple output (MIMO) systems draw much attention from the military circles. This is because these systems offer a high potential reliability. Modeling of MIMO is, however, extremely complex. For example, Jootar et al. [1] use the residue theorem to define the probability of error for a simple 2-*nd* order system

$$P_b = \frac{1}{2\pi j}\int_{-\infty+je}^{\infty+je} \frac{1}{v\prod_1^4 (1-jv\lambda_i)} dv = -Res[\varphi_z(s)/s \; at \; \mathrm{LHP} \;\; poles] \tag{1}$$

Alternatively, P_b can be found numerically by the Gauss-Chebyshev approximation [2] or via simulation methods [3], [4].

In this paper the simple closed-form expressions are derived for a wide class of MIMO systems on the basis of positional statistics.

2. Assumptions and equations

Let us consider the set of N independent objects, each described by the same kind of the density distribution function $f(\Gamma)$. Using this function one can arrange a new set of ordered sequences, $m = 1, 2,\ldots,N$. The first sequence denoted by $m = 1$ is then assigned by the lowest possible values of Γ, the next one of $m = 2$ – by the second lowest values of Γ and so on. The general formula for m-th sequence is [5]

$$f_m(\Gamma \mid N) = \frac{N!f(\Gamma)}{(m-1)!(N-m)!} F^{m-1}(\Gamma)[1-F(\Gamma)]^{N-m} \tag{2}$$

where *f(Γ), F(Γ)* are density- and cumulative distribution functions (*ddf, cdf*), respectively.

Let *Γ* mean the signal-to-noise power ratio (SNR) in a set of channels subjected to flat and slow Rayleigh fading. The *ddf* for a single channel is then as follows [6]

$$f(\gamma) = \gamma_0 e^{-\gamma/\gamma_0} \tag{3}$$

where γ_0 is a mean value of signal-to-noise ratio.

The *cdf* – being the integral of *ddf* – takes then a form

$$F(\gamma) = 1 - e^{-\gamma/\gamma_0} \tag{4}$$

Putting (3) and (4) into (2) one obtain

$$f_m(\gamma \mid N, \gamma_0) = \frac{N!}{(m-1)!(N-m)!} \gamma_0^{-1} e^{-\gamma/\gamma_0} \left(1 - e^{-\gamma/\gamma_0}\right)^{m-1} \left(e^{-\gamma/\gamma_0}\right)^{N-m} \tag{5}$$

From the viewpoint of bit error rate (BER) the most interesting is the sequence of *m* = *N*, which represents the virtual channel of the highest values of *Γ*. This corresponds to data of the optimal switching diversity system. The equation (5) for *m* = *N* takes the form

$$f_N(\gamma \mid \gamma_0) = \frac{N}{\gamma_0} e^{-\gamma/\gamma_0} (1 - e^{-\gamma/\gamma_0})^{N-1} \tag{6}$$

Since the changes of *γ* are assumed slow, the resultant BER for the best channel can be calculated as the error probability *P(*γ) for no fading conditions averaged over the density distribution function (6)

$$P_N(\gamma_0) = \int_0^{\infty} P(\gamma) f_N(\gamma \mid \gamma_0) d\gamma \tag{7}$$

The probability of error for the differential phase shift keying mode, DPSK, and AWGN is

$$P(\gamma) = 1/2e^{-\gamma} \tag{8}$$

Putting (8) and (6) into (7) we finally obtain

$$P_N(\gamma_0) = \frac{N}{2\gamma_0} \int_0^{\infty} e^{-\gamma} e^{-\gamma/\gamma_0} (1 - e^{-\gamma/\gamma_0})^{N-1} d\gamma = \frac{N}{2} \sum_{k=0}^{N-1} (-1)^k \binom{N-1}{k} \frac{1}{\gamma_0 + k + 1} \tag{9}$$

3. BER curves

The equation (9) defines the probability of error (BER) for MIMO systems with DPSK modulation and switching diversity mode. The variables are: the mean signal-to-noise power ratio γ_0 and the diversity order N. In the sequel we will consider $N = 1, 2, 4$ and 6. They respond to some real antennae arrangements [4]. The appropriate BER formulae are then due to (9) as follows

$$P_1(\gamma_0) = 1/(2\gamma_0 + 2) \tag{10}$$

$$P_2(\gamma_0) = \frac{1}{\gamma^2 + 3\gamma + +2} \tag{11}$$

$$P_4(\gamma_0) = \frac{12}{\gamma^4 + 10\gamma^3 + 35\gamma^2 + 50\gamma + 24} \tag{12}$$

$$P_6(\gamma_0) = \frac{360}{\gamma^6 + 21\gamma^5 + 175\gamma^4 + 735\gamma^3 + 1624\gamma^2 + 1764\gamma + 720} \tag{13}$$

The diagrams of (10÷13) are depicted in Fig. 1. One can observe the regularity of curves and highly decreasing BER along with the diversity order N and SNR.

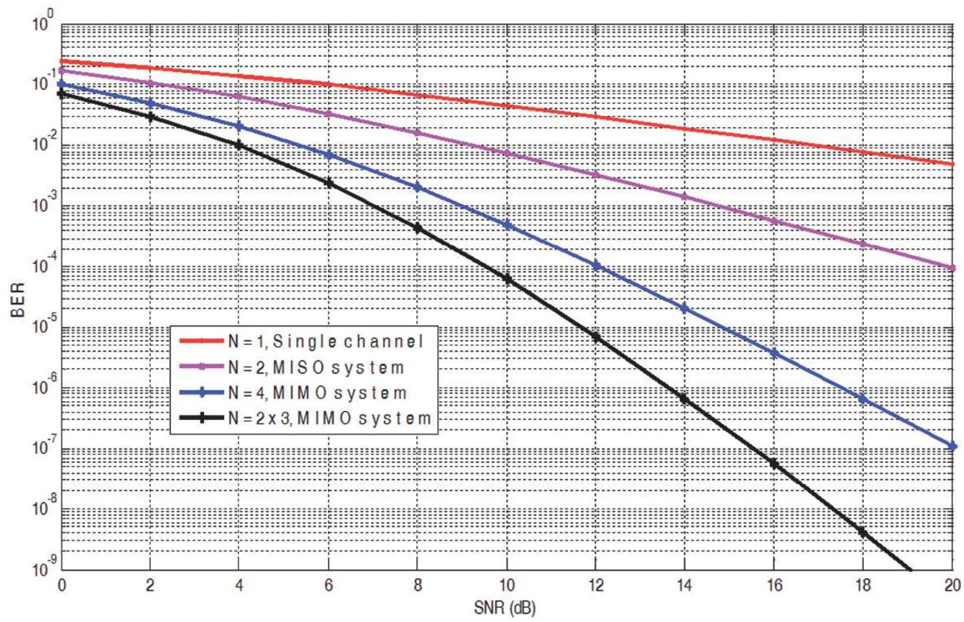

Figure 1. Original BER curves for *N-th* order switching diversity DPSK MIMO system

4. Comparison with other approaches

The switching diversity exploits the best channel to detect the useful signal. The more often used, at least in the theory, is the maximum ratio combining (MRC), which exploits all the channels to detect this signal. Also, the coherent phase shift keying, PSK, is often used as a reference instead of DPSK. These differences may affect the BER characteristics.

Let us consider first the consequence of the keying mode, PSK – DPSK. The known P_b formula for Rayleigh PSK channel is [7]

$$P_{PP}(\gamma) = 0.5\left(1 - \sqrt{\frac{\gamma}{\gamma+2}}\right) \tag{14}$$

It can be shown on the basis of series theory that this formula approaches eq. (10) for $\gamma \to \infty$ (and for $\gamma \to 0$). Meanwhile the differences are done by the following expression (see Tab. 1)

$$\Delta P_{PP} = \sqrt{\frac{\gamma}{\gamma+2}} - \frac{\gamma}{\gamma+1} \tag{15}$$

Table 1. BER differences between eq. (10) and (14)

γ [W|dB]	0|-∞	1| 0	10|10	100|20
ΔP_{PP} [%]	0	8.47	0.286	0.0035

The formulae (10) and (14) are depicted in Fig. 2. One can see that a difference in SNR is of the order of 0÷1 dB in favor of PSK and it is observed mainly around the point $\gamma \approx 1$(0dB).

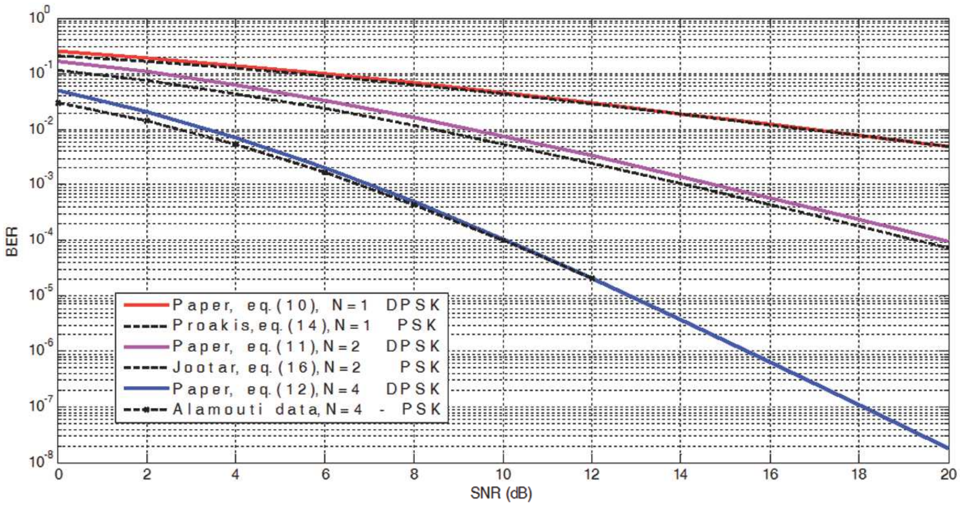

Figure 2. A comparison of formulae (10)÷(13) with results of other authors

Let us consider further the consequences of the switching/combining diversity and the PSK/DPSK modes. The approximation formula of Jootar et al. for MRC-PSK system of $N = 2$ is as follows [1]

$$P_{JP} \approx \frac{1}{4}\left(2+\sqrt{\frac{\gamma}{2\gamma+2}}\right)\left(1-\sqrt{\frac{\gamma}{2\gamma+2}}\right)^2 \tag{16}$$

The appropriate equation for switching diversity and DPSK is given by eq.(11). Both formulae, (11) and (16), are depicted in Fig. 2. One can see that a difference in SNR reaches 1÷2 dB in favor of MRC and PSK.

The closed-form expressions for systems of $N \geq 4$ are not known to the author. Hence, the Alamouti simulation data for MRC-PSK and $N = 4$ has been taken as the reference [3]. In the simulation process, however, each antenna generates additional power, which is not the case in the analysis. Then, some authors use the following correction [4]

$$\overline{\gamma} = 10^{\,n/10}\gamma \tag{17}$$

where n – number of receiving antennas.

Using $n = 2$ and putting $\overline{\gamma}$ instead of γ in (12) one obtains the analytical curve for the MIMO system of $N = 4$, Fig. 2 (solid line). There is also shown the Alamouti curve [3] (dotted line). One can see that an agreement is fully satisfied.

5. Conclusion

The simple closed-form expressions for bit error rate (BER) in switching diversity MIMO systems subjected to flat and quasi-static Rayleigh fading have been derived. This derivation is based on the positional statistics and is extremely simple.

The agreement of the obtained results with the data of other authors – using another diversity modes and/or more complex and/or approximated methods – is satisfied. The further development of the presented approach for the dynamic fading is provided for [4,8].

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